where

$$L_x=(\alpha+\beta)I-(\beta-\alpha)\frac{p}{2}\,\mu_x\delta_x I+(\alpha+\beta)\frac{p}{2}\,\mu_x\delta_x\mathbf{A}^n-(\beta-\alpha)\,\frac{p^2}{4}\delta_x^2\mathbf{A}^n$$

$$L_y=(\alpha+\beta)I-(\beta-\alpha)\frac{p}{2}\mu_y\delta_y I+(\alpha+\beta)\frac{p}{2}\mu_y\delta_y\mathbf{B}^n-(\beta-\alpha)\,\frac{p^2}{4}\delta_y^2\mathbf{B}^n$$

$\mathbf{A}=\dfrac{\partial \mathbf{f}}{\partial \mathbf{u}}$, $\mathbf{B}=\dfrac{\partial \mathbf{g}}{\partial \mathbf{u}}$, α and β are arbitrary parameters.

Beam-Warming method

$$\left(\mathbf{I}+\frac{p}{2}\mu_y\delta_y\mathbf{B}^n\right)\left(\mathbf{I}+\frac{p}{2}\mu_x\delta_x\mathbf{A}^n\right)\mathbf{u}^{n+1}$$

$$=\left(\mathbf{I}+\frac{p}{2}\mu_y\delta_y\mathbf{B}^n\right)\left(\mathbf{I}+\frac{p}{2}\mu_x\delta_x\mathbf{A}^n\right)\mathbf{u}^n-p(\mu_x\delta_x\mathbf{f}^n+\mu_y\delta_y\mathbf{g}^n) \qquad (6.183)$$

Here, we have chosen $\alpha=\beta$ in (6.182).

Example 6.6 Solve the initial boundary value problem

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{4}\right)+\frac{\partial}{\partial y}\left(\frac{u^2}{4}\right)=0$$

$$u(x, y, 0)=\frac{1}{2}(x+y)^2 \qquad 0\leqslant x, y\leqslant 1$$

$$u(0, y, t)=\left(\frac{1-(1+yt)^{1/2}}{t}\right)^2$$

$$u(x, 0, t)=\left(\frac{1-(1+xt)^{1/2}}{t}\right)^2$$

using the MacCormack method with $h=1/3$ and $p=1/2$.

The nodal points are

$$x_l=lh, \qquad 0\leqslant l\leqslant 3$$
$$y_m=mh, \qquad 0\leqslant m\leqslant 3$$
$$t_n=nk, \qquad n=0, 1, 2, \ldots$$

The initial and boundary conditions become

$$u_{l,m}^0=\frac{1}{2}(l+m)^2h^2 \qquad 0\leqslant l, m\leqslant 3$$

$$u_{0,m}^n=\left(\frac{1-(1+mnhk)^{1/2}}{nk}\right)^2, \quad n=1, 2, \ldots, 0\leqslant m\leqslant 3$$

$$u_{l,0}^n=\left(\frac{1-(1+nlhk)^{1/2}}{nk}\right)^2, \quad n=1, 2, \ldots, 0\leqslant l\leqslant 3$$

TABLE 6.6 MAXIMUM ABSOLUTE ERROR (ERROR × 10^6) AFTER 50 TIMES STEPS ON LINES PARALLEL TO x-AXIS, $h=0.1$, $\gamma=\beta/\alpha$

p	1.0							4.0						8.0					
γ	1.0	6.0	6.5	6.8	7.0	7.5	8.0	1.0	1.4	1.42	1.44	1.46	1.5	1.0	1.1	1.14	1.16	11.8	1.2
y																			
0.2	433	92	71	60	53	36	32	308	76	56	36	38	47	225	167	111	76	40	18
0.4	406	65	46	35	33	45	56	246	34	28	38	56	77	171	111	64	36	23	45
0.6	346	42	36	45	51	64	75	212	42	55	67	80	106	136	76	32	29	47	71
0.8	354	42	57	65	71	83	94	189	57	71	85	100	129	115	49	18	41	67	95
1.0	365	63	78	86	91	103	113	151	71	85	99	114	143	89	24	28	53	80	109

TABLE 6.7 MAXIMUM ABSOLUTE ERROR (ERROR × 10^6) AFTER 300 TIME STEPS ON LINES PARALLEL TO x-AXIS, $h=0.1$, $\gamma=\beta/\alpha$

p	1.0					4.0					8.0			
γ	1.0	6.0	6.5	7.0	7.5	1.0	1.4	1.42	1.45	1.5	1.0	1.15	1.18	1.2
y														
0.2	260	60	48	37	28	120	32	25	14	5	79	31	15	4
0.4	207	40	31	23	16	97	23	17	8	8	63	25	12	2
0.6	175	30	23	16	10	80	17	12	4	10	53	21	8	0.9
0.8	161	23	17	11	5	69	13	9	2	11	47	17	6	2
1.0	142	18	12	7	2	62	10	6	0.8	12	42	14	4	4

TABLE 6.8 ERROR × 10^6 AT THE CENTRAL GRID POINT, $h=0.1$, $\gamma = \beta/\alpha$

	$p=1.0$					$p=4.0$				$p=8.0$		
Time steps / γ	5.0	6.0	6.5	7.0	7.5	1.4	1.42	1.45	1.5	1.15	1.18	1.2
50	−71	−30	−12	3	18	−5	11	36	80	−28	17	50
300	−52	−33	−25	−18	−11	−20	−14	−6	9	−23	−10	−0.6

central grid point. Accurate results are obtained for $p=1$, 4 and 8 when β/α lies in the range $6.0 \leqslant \beta/\alpha \leqslant 8.0$, $1.3 \leqslant \beta/\alpha \leqslant 1.6$ and $1.1 \leqslant \beta/\alpha \leqslant 1.25$ respectively. It is seen that for fixed p, a value of β/α in the given range can be found which has an accuracy better than the results given in Table 6.8. The errors in the solution using the Beam-Warming method ($\alpha=\beta$), are higher than the results obtained here.

Bibliographical Note

The excellent texts dealing with the numerical solutions of the hyperbolic equations are 9, 96, 184 and 203. The stability of the linear finite difference equations is discussed in 168. The high order difference schemes are given in 80, 126 and 129. The difference schemes for the second order hyperbolic differential equations with variable coefficients and with or without mixed derivatives are studied in 44, 169, 180 and 181. The solution of one dimensional wave equation under derivative boundary conditions has been examined in 150.

The LOD method for obtaining the numerical solution of the hyperbolic equations in two and three space dimensions is given in 98, 130 and 215.

The explicit and implicit difference schemes for the system of hyperbolic equations are discussed in 2, 96, 99, 116, 159, 167, 176, 178, 179, 202, 204, 210, 225, 232 and 239. The *Kreiss* stability analysis of the difference schemes is given in 1, 16, 90, 95, 105 and 158.

I roblems

1. The function $u(x, t)$ satisfies the differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + cu$$

with boundary conditions

$$u=0 \text{ for } x=0 \text{ and } x=1,\ t \geqslant 0$$

Let u and $\partial u/\partial t$ be prescribed for $t=0$, $0 \leqslant x \leqslant 1$.

(i) Derive the difference scheme by replacing the derivatives by central differences.
(ii) Obtain the principal part of the truncation error.
(iii) Determine the stability criterion of the difference scheme.

2. The differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + cu$$

is approximated by the difference scheme

$$(1+\tau\delta_t^2)^{-1}\delta_t^2 u_m^n = p^2\delta_x^2 u_m^n + cp^2h^2u_m^n$$

where τ is arbitrary, $p=k/h$ and c is a constant.

Use the explicit method

$$u_m^{n+1} = 2(1-p^2)u_m^n + p^2(u_{m-1}^n + u_{m+1}^n) - u_m^{n-1}$$

and central difference approximation for the derivative conditions, to calculate a solution for $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 0.5$ with $h=k=.1$.

8. The first and second *Lees* ADI methods for solving the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

can be written as

(i) $u^{*n+1}_{l,m} = 2u^n_{l,m} - u^{n-1}_{l,m} + p^2\delta_x^2[\eta u^{*n+1}_{l,m} + (1-2\eta)u^n_{l,m} + \eta u^{n-1}_{l,m}]$

$\quad + p^2\delta_y^2[(1-2\eta)u^n_{l,m} + 2\eta u^{n-1}_{l,m}]$

$u^{n+1}_{l,m} = u^{*n+1}_{l,m} + p^2\eta\delta_y^2(u^{n+1}_{l,m} - u^{n-1}_{l,m})$

and

(ii) $u^{*n+1}_{l,m} = 2u^n_{l,m} - u^{n-1}_{l,m} + p^2\delta_x^2[\eta u^{*n+1}_{l,m} + (1-2\eta)u^n_{l,m} + \eta u^{n-1}_{l,m}] + p^2\delta_y^2\, u^n_{l,m}$

$u^{n+1}_{l,m} = u^{*n+1}_{l,m} + \eta p^2\delta_y^2(u^{n+1}_{l,m} - 2u^n_{l,m} + u^{n-1}_{l,m})$

where η is arbitrary.

Determine the uniform difference schemes in (i) and (ii). Show that the principal parts of the truncation error and the stability criteria are the same for both methods.

9. Write the first and second one parameter *Lees* ADI methods for the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + cu$$

Determine the order of accuracy and the stability criterion ror both methods.

10. The first and second *Lees* ADI methods for the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

are of the form

(i) $u^{*n+1} = 2u^n - u^{n-1} + p^2\delta_x^2[\eta u^{*n+1} + (1-2\eta)u^n + \eta u^{n-1}]$

$\quad + p^2(\delta_y^2 + \delta_z^2)[(1-2\eta)u^n + 2\eta u^{n-1}]$

$u^{**n+1} = u^{*n+1} + p^2\eta\delta_y^2(u^{**n+1} - u^{n-1})$

$u^{n+1} = u^{**n+1} + p^2\eta\delta_z^2(u^{n+1} - u^{n-1})$

and

(ii) $u^{*n+1} = 2u^n - u^{n-1} + p^2\delta_x^2[\eta u^{*n+1} + (1-2\eta)u^n + \eta u^{n-1}] - p^2(\delta_y^2 + \delta_z^2)u^n$

$u^{**n+1} = u^{*n+1} + \eta p^2\delta_y^2(u^{**n+1} - 2u^n + u^{n-1})$

$u^{n+1} = u^{**n+1} + \eta p^2\delta_z^2(u^{n+1} - 2u^n + u^{n-1})$